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STRONGLY MEAGER SETS CAN BE QUITE BIG

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ABSTRACT

Assume **CH**. There exists a strongly meager set $X \subseteq 2^\omega$ and a continuous function $F\colon 2^\omega \longrightarrow 2^\omega$ such that $F"(X) = 2^\omega$. The analogous statement for the strong measure zero, the notion dual to strongly meager, is false.

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1. Introduction

We say that a set $X \subseteq 2^\omega$ is strongly measure zero ($X \in \mathcal{SN}$) if for every meager set $F \in \mathcal{M}$, $X + F \neq 2^\omega$. A set $X \subseteq 2^\omega$ is strongly meager ($X \in \mathcal{SM}$) if for every null set $H \in \mathcal{N}$, $X + H \neq 2^\omega$.

In spite of having similar definitions, strongly measure zero sets and strongly meager sets behave quite differently. The most illustrative example of this asymmetry is the fact that $\mathcal{SN}$ sets form a σ-ideal, while under CH the union of two $\mathcal{SM}$ sets doesn't have to be a member of $\mathcal{SM}$ ([2]). On the other hand, strongly meager sets have certain combinatorial properties which are not necessarily satisfied by strongly measure zero sets. For instance, $\mathcal{SM}$ sets are completly Ramsey null ([4]), while under CH there are $\mathcal{SN}$ sets that do not have this property ([5]). In the main theorem of this paper we show that in contrast with $\mathcal{SN}$ sets, it is possible (assuming CH) to find $X \in \mathcal{SM}$, and a continuous function $F\colon 2^\omega \to 2^\omega$ which maps X onto 2^ω. Before we start, let us recall a couple of standard definitions and basic facts from the theory of small subsets of 2^ω.

We define $\mathcal{I} = \{X \subseteq 2^\omega\colon \forall F\colon 2^\omega \longrightarrow 2^\omega \text{ continuous}, F"(X) \neq 2^\omega\}$ and we denote by $(s)_0$ the ideal of Marczewski sets, that is

$$(s)_0 = \{X \subseteq 2^\omega\colon \forall P \text{ perfect subset of } 2^\omega \exists P' \text{ a perfect subset of } P \text{ so that } X \cap P' = \emptyset\}.$$

The following is well-known:

LEMMA 1:
(1) *$\mathcal{I}$ is a σ-ideal,*
(2) *$\mathcal{I} \subseteq (s)_0$,*
(3) *$\mathcal{I} \neq (s)_0$ (in ZFC).*

Notice that such a σ-ideal was defined and investigated in several papers; see, for example, [3].

Since strongly meager sets and strong measure zero sets are $(s)_0$, it makes sense to ask if they are in $\mathcal{I}$.

It is well-known that $\mathcal{SN} \subseteq \mathcal{I}$. In fact, if $F\colon 2^\omega \longrightarrow 2^\omega$ is a continuous function and $X \in \mathcal{SN}$, then $F"(X) \in \mathcal{SN}$.

As we mentioned before, the purpose of this paper is to show:

THEOREM 2: *It is consistent with ZFC that $\mathcal{SM} \not\subseteq \mathcal{I}$.*

Note that it is easy to see (under CH, for example) that $\mathcal{I} \not\subseteq \mathcal{SM}$.

2. Combinatorics

The following theorem is the finitary version of the construction.

THEOREM 3: *For every $k \in \omega$, $\varepsilon, \delta, \epsilon > 0$ there exists $n \in \omega$ such that if $I \subseteq \omega$, $|I| > n$ then there is a partition $2^I = a^0 \dot\cup a^1$ such that*

(1) *if $X \subseteq 2^I$, $|X| \leq k$, then $\left| \frac{|\bigcap_{x \in X}(a^0+x)|}{2^{|I|}} - \frac{1}{2^{|X|}} \right| < \delta$;*

(2) *if $U \subseteq 2^I$, and $\frac{|U|}{2^{|I|}} = a \geq \varepsilon$, then there exists a set $T_U \subseteq 2^I$, $\frac{|T_U|}{2^{|I|}} > 1 - \epsilon$ such that for every $s \in T_U$, $\left| \frac{|(a^0+s) \cap U|}{2^{|I|}} - \frac{a}{2} \right| < \epsilon$.*

Proof: (1) This is a special case of a result proved in [2] and we will need it only for $k = 2$. Fix k, ε, δ, and choose the set $C \subseteq 2^I$ randomly (for the moment I is arbitrary). For each $s \in 2^I$, decisions whether $s \in C$ are made independently with the probability of $s \in C$ equal to $1/2$. Thus the set C is a result of a sequence of Bernoulli trials. Note that by Chebyshev's inequality, the probability that $1/2 + \delta \geq |C| \cdot 2^{-|I|} \geq 1/2 - \delta$ approaches 1 as $|I|$ goes to infinity.

Let S_n be the number of successes in n independent Bernoulli trials with probability of success p. We will need the following well-known fact.

THEOREM 4 (Bernstein's Inequality): *For every $\delta > 0$,*

$$P\Big(|\frac{S_n}{n} - p| \geq \delta\Big) \leq 2e^{-n\delta^2/4}.$$

Consider an arbitrary set $X \subseteq 2^I$. To simplify the notation denote $V = 2^I \setminus C$ and note that $\bigcap_{s \in X}(C+s) = 2^I \setminus (V + X)$. For a point $t \in 2^I$, $t \notin X + V$ is equivalent to $(t + X) \cap V = \emptyset$. Thus the probability that $t \notin X + V$ is equal to $2^{-|X|}$, as $t \notin X + V$ means that $t + x \notin V$ for $x \in X$.

Let $G(X)$ be a subgroup of $(2^I, +)$ generated by X. Since every element of 2^I has order 2, it follows that $|G(X)| \leq 2^{|X|}$.

LEMMA 5: *There are sets $\{U_j : j \leq |G(X)|\}$ such that:*

(1) $\forall j \forall s, t \in U_j \ (s \neq t \rightarrow s + t \notin G(X))$,

(2) $\forall j \leq |G(X)| |U_j| = 2^{|I|}/|G(X)|$,

(3) $\forall i \neq j U_i \cap U_j = \emptyset$,

(4) $\bigcup_{j \leq |G(X)|} U_j = 2^I$.

Proof: Choose U_j's to be disjoint selectors from the cosets $2^I/G(X)$. ∎

Note that if $t_1, t_2 \in U_j$ then the events $t_1 \in X + V$ and $t_2 \in X + V$ are independent, since sets $t_1 + X$ and $t_2 + X$ are disjoint. Consider the sets $X_j =$

$U_j \cap \bigcap_{s\in X}(C+s)$, for $j \leq |G(X)|$. The expected value of the size of this set is $2^{-|X|} \cdot 2^{|I|}/|G(X)|$. By Theorem 4, for each $j \leq |G(X)|$,

$$P\Big(\Big|\frac{|X_j|}{2^{|I|}/|G(X)|} - 2^{-|X|}\Big| \geq \delta\Big) \leq 2e^{-2^{|I|-2}\delta^2/|G(X)|}.$$

It follows that for every $X \subseteq 2^I$ the probability that

$$2^{-|X|} - \delta \leq \frac{|\bigcap_{s\in X}(C+s)|}{2^{|I|}} \leq 2^{-|X|} + \delta$$

is at least

$$1 - 2|G(X)|e^{-2^{|I|-2}\delta^2/|G(X)|} \geq 1 - 2^{|X|+1}e^{-2^{|I|-|X|-2}\delta^2}.$$

The probability that it happens for every X of size $\leq k$ is at least

$$1 - 2^{|I|\cdot(k+1)^2} \cdot e^{-2^{|I|-k-2}\delta^2}.$$

If k and δ are fixed, then this expression approaches 1 as $|I|$ goes to infinity, since $\lim_{x\to\infty} P(x)e^{-x} = 0$ for any polynomial $P(x)$. It follows that for sufficiently large $|I|$ the probability that the "random" set C has the required properties is > 0. Thus there exist an actual $C = a^0$ and $2^I \setminus C = a^1$ with these properties as well.

(2) Let a^0 be the set constructed above; without loss of generality we can assume that

$$\frac{|a^0|}{2^{|I|}} = \frac{1}{2}.$$

Define $A = \{(s,t) : t \in a^0 + s\}$. Note that A is symmetric, that is $(A)_s = \{t : (s,t) \in A\} = (A)^s = \{t : (t,s) \in A\}$.

Let

$$U \subseteq 2^I \quad \text{and} \quad \frac{|U|}{2^{|I|}} = a \geq \varepsilon.$$

We want to know how many vertical sections of the set $A \cap (2^I \times U)$ are of size approximately $1/2$ relative to U. For $s \in U$, consider random variables $X_s = \chi((A)^s)$, where $\chi((A)^s)$ is the characteristic function of $(A)^s$. Note that the expected value $\mathbf{E}(X_s) = \frac{1}{2}$. Moreover,

$$\mathbf{E}\Big(\Big(X_s - \frac{1}{2}\Big)^2\Big) = \frac{1}{4}$$

and for $s \neq t$,

$$\mathbf{E}\Big(\Big|\Big(X_s - \frac{1}{2}\Big)\Big(X_t - \frac{1}{2}\Big)\Big|\Big) = \mathbf{E}\Big(\Big|X_sX_t - \frac{1}{4}\Big|\Big) \leq \delta.$$

Note that

$$\begin{aligned}&\left|\left\{s \in 2^I : \left|\frac{(a^0+s)\cap U}{2^{|I|}} - \frac{a}{2}\right| > \epsilon\right\}\right|\\ &\quad = \left|\left\{s \in 2^I : \left|\frac{(a^0+s)\cap U}{|U|}\frac{|U|}{2^{|I|}} - \frac{1}{2}\cdot\frac{|U|}{2^{|I|}}\right| > \epsilon\right\}\right|\\ &\quad \leq \left|\left\{s \in 2^I : \left|\frac{(a^0+s)\cap U}{|U|} - \frac{1}{2}\right| > \frac{\epsilon}{\varepsilon}\right\}\right|.\end{aligned}$$

Therefore it suffices to estimate

$$P\Big(\Big|\frac{\sum_{s\in U} X_s}{|U|} - \frac{1}{2}\Big| > \frac{\epsilon}{\varepsilon}\Big).$$

Note that by Chebyshev's inequality

$$\begin{aligned}P\Big(\Big|\frac{\sum_{s\in U} X_s}{|U|} - \frac{1}{2}\Big| > \frac{\epsilon}{\varepsilon}\Big) &= P\Big(\Big|\sum_{s\in U}(X_s - \frac{1}{2})\Big| > |U|\frac{\epsilon}{\varepsilon}\Big)\\ &\leq \frac{\varepsilon^2}{|U|^2\epsilon^2}\mathbf{D}^2\Big(\sum_{s\in U}\Big(X_s - \frac{1}{2}\Big)\Big) = \frac{\varepsilon^2}{|U|^2\epsilon^2}\mathbf{E}\Big(\Big(\sum_{s\in U}\Big(X_s - \frac{1}{2}\Big)\Big)^2\Big).\end{aligned}$$

Finally observe that

$$\begin{aligned}&\frac{\varepsilon^2}{|U|^2\epsilon^2}\mathbf{E}\Big(\Big(\sum_{s\in U} X_s - \frac{1}{2}\Big)^2\Big)\\ &\quad = \frac{\varepsilon^2}{|U|^2\epsilon^2}\mathbf{E}\Big(\sum_{s\in U}\Big(X_s - \frac{1}{2}\Big)^2 + \sum_{s,t\in U, s\neq t}\Big(X_s - \frac{1}{2}\Big)\Big(X_t - \frac{1}{2}\Big)\Big)\\ &\quad \leq \frac{\varepsilon^2}{|U|^2\epsilon^2}\Big(\frac{|U|}{4} + \delta\frac{|U|(|U|-1)}{2}\Big) \leq \frac{\varepsilon^2}{4\epsilon^2}\cdot\frac{1}{|U|} + \frac{\delta\varepsilon^2}{2\epsilon^2}\frac{|U|-1}{|U|}.\end{aligned}$$

In (1) we showed that as $n \to \infty$ then $\delta \to 0$. Therefore if ε, ϵ are fixed, a large value of $|I|$ will result in

$$\frac{\varepsilon^2}{4\epsilon^2}\cdot\frac{1}{|U|} + \frac{\delta\varepsilon^2}{2\epsilon^2}\frac{|U|-1}{|U|} < \epsilon,$$

which means that

$$\frac{|\{s \in 2^I : |\frac{(a^0+s)\cap U}{2^{|I|}} - \frac{a}{2}| \leq \epsilon\}|}{2^{|I|}} \geq P\Big(\Big|\frac{\sum_{s\in U} X_s}{|U|} - \frac{1}{2}\Big| \leq \frac{\epsilon}{\varepsilon}\Big) \geq 1 - \epsilon.$$

Finally let

$$T_U = \Big\{s \in 2^I : \Big|\frac{(a^0+s)\cap U}{2^{|I|}} - \frac{a}{2}\Big| \leq \epsilon\Big\}.$$

Note that ϵ is the only independent parameter. That is given ε we can take $\epsilon < \varepsilon^2, k = 2$ and $|I|$ so large that the corresponding δ is small enough for the approximations to go through. ∎

Definition 1 ([2]): Suppose that I is a finite set. A distribution is a function $m: I \longrightarrow \mathbb{R}$ such that

$$0 \leq m(x) \leq \frac{1}{|I|}.$$

Let $\overline{m} = \sum_{s \in I} m(s)$.

To illustrate this concept suppose that $A \subseteq 2^\omega$ is a measurable set and $n \in \omega$. Define m on 2^n by $m(s) = \mu(A \cap [s])$, for $s \in 2^n$. A specific instance of this definition that we will use often in the sequel is when A is a clopen set. In particular, if $K = I_1 \cup I_2$ and $J \subseteq 2^K$, define distribution m on 2^{I_1} as follows: for $s \in 2^{I_1}$, let

$$m(s) = \frac{|\{t \in J : s \subseteq t\}|}{2^{|K|}}.$$

The following theorem is an extension of Theorem 3 dealing with distributions instead of sets.

THEOREM 7: *For every $\epsilon, \varepsilon > 0$, there exists $n \in \omega$ such that if $I \subseteq \omega$, $|I| > n$, then there is a partition $2^I = a^0 \cup a^1$ such that if m is a distribution on 2^I, and $\overline{m} \geq \varepsilon$, then there exists a set $T_m \subseteq 2^I$, $|T_m|/2^{|I|} > 1 - \epsilon$ such that for every $s \in T_m$, $|\sum\{m(t) : t \in a^0 + s\} - \overline{m}/2| < \epsilon$.*

Proof: Suppose that $\varepsilon^2 > \epsilon$ are given and $a^0 \dot\cup a^1 = 2^I$ are as in Theorem 3, for $\varepsilon' = \epsilon^2$ and $\epsilon' = \epsilon^3$. First observe that if

$$m = \frac{b}{2^{|I|}} \cdot \chi_U,$$

where $U \subseteq 2^I$ and $|U|/2^{|I|} = a \geq \epsilon^2$, $0 < b \leq 1$ and χ_U is a characteristic function of the set U, then it follows immediately from Theorem 3 that for $s \in T_U$,

$$\left| \sum\{m(t) : t \in a^0 + s\} - \overline{m}/2 \right| < b\epsilon^3 \leq \epsilon.$$

Next, note that if $\{U_i : i < \ell\}, \{b_i : i < \ell\}$ are such that

(1) $U_i \cap U_j = \emptyset$, for $i \neq j$,
(2) $0 < b_i \leq 1$, for $i < \ell$,
(3) $|U_i|/2^{|I|} \geq \epsilon^2$, for every $i < \ell$,
(4) $m_i = \frac{b_i}{2^{|I|}} \cdot \chi_{U_i}$,

then for $m = \sum_{i<\ell} m_i$ and $s \in \bigcap_{i<\ell} T_{U_i}$, we have

$$\left|\sum\{m(t) : t \in a^0 + s\} - \overline{m}/2\right| < \sum_{i<\ell} b_i\epsilon^3 \leq \ell\epsilon^2 \quad \text{and} \quad \frac{|\bigcap_{i<\ell} T_{U_i}|}{2^{|I|}} \geq 1 - \ell\epsilon^2.$$

Consider an arbitrary distribution m with $\overline{m} \geq \epsilon^2$ and let $\ell = 1/\epsilon$ (without loss of generality it is an integer). Let $\{U_i : i < \ell\}$ be defined as

$$U_i = \{s \in 2^I : i\epsilon/2^{|I|} < m(s) \leq (i+1)\epsilon/2^{|I|}\}.$$

Let $K = \{i : |U_i|/2^{|I|} \geq \epsilon^2\}$. Put

$$m' = \sum_{i \in K} \frac{i\epsilon}{2^{|I|}} \cdot \chi_{U_i} \quad \text{and} \quad U = \bigcup_{i<\ell} U_i.$$

Note that for $s \in U$, and $i \in K$, $m(s) - m'(s) \leq \epsilon/2^{|I|}$ and $|\overline{m} - \overline{m'}| \leq \epsilon + \sum_{i<\ell} \epsilon^2 = 2\epsilon$.

Apply 3 to each of the sets $\{U_i : i \in K\}$ to get sets T_{U_i} and put $T_m = \bigcap_{i\in K} T_{U_i}$. Clearly $|T_m|/2^{|I|} \geq 1 - \ell\epsilon^3 = 1 - \epsilon^2$. Now, for $t \in T_m$,

$$\sum\{m(s): s \in a^0+t\} \leq \sum\{m'(s): s \in a^0+t\}+2\epsilon \leq \overline{m'}/2+\epsilon^3+2\epsilon+\epsilon^2 \leq \overline{m}/2+3\epsilon.$$

The lower estimate is the same and we get

$$|\sum\{m(s) : s \in a^0 + t\} - \overline{m}/2| \leq 3\epsilon. \quad \blacksquare$$

3. ZFC result

As a warm-up before proving the main result we will show a ZFC result that uses only a small portion of the combinatorial tools developed above.

In order to show that $\mathcal{SN} \subseteq \mathcal{I}$ one could use the following result:

THEOREM 8 ([4]): *Suppose that $F: 2^\omega \longrightarrow 2^\omega$ is a continuous function. There exists a set $H \in \mathcal{M}$ such that*

$$\forall z \in 2^\omega \exists y \in 2^\omega F^{-1}(y) \subseteq H + z.$$

We will show that the measure analog of this theorem is false.

THEOREM 9: *There exists a continuous function $F: 2^\omega \longrightarrow 2^\omega$ such that for every set $G \in \mathcal{N}$,*

$$\{z : \exists y F^{-1}(y) \subseteq G + z\} \in \mathcal{N}.$$

Proof: Let δ_n, k_n, I_n for $n \in \omega$ be such that

(1) $\delta_n = 4^{-n-3}$, $k_n = 2^{n+3}$,

(2) I_n is chosen as in Theorem 3(1), for $k = k_n$ and $\delta = \delta_n$ and

(a) $|I_{n+1}| \geq 4^{n+10} 2^{|I_0 \cup \cdots \cup I_n|}$,

(b) $\min(I_{n+1}) \geq \max(I_n)$.

Let $2^{I_n} = a_n^0 \dot{\cup} a_n^1$ be a partition as in Theorem 3(1).

Define $F: 2^\omega \longrightarrow 2^\omega$ as

$$F(x)(n) = i \iff x{\restriction}I_n \in a_n^i.$$

Note that for every $x \in 2^\omega, F^{-1}(x) = \prod_n a_n^{x(n)}$ is a perfect set.

Suppose that $G \subseteq 2^\omega$ is null, and let $U \subseteq 2^\omega$ be an open set of measure $1/2 > \varepsilon > 0$ containing G. We will show that

$$\mu(\{z : \exists y F^{-1}(y) \subseteq U + z\}) \longrightarrow 0 \quad \text{as } \varepsilon \to 0.$$

Let

$$U_0^\emptyset = \left\{ s \in 2^{I_0} : \frac{\mu([s] \cap U)}{\mu([s])} \geq \frac{3}{4} \right\},$$

and for $n > 0$ and $t \in 2^{I_0 \cup \cdots \cup I_{n-1}}$, let

$$U_n^t = \left\{ s \in 2^{I_n} : \frac{\mu([t^\frown s] \cap U)}{\mu([t^\frown s])} \geq 1 - \frac{1}{2^{n+2}} \right\}.$$

Easy computation shows that $\mu(U_0^\emptyset) \leq 4/3\varepsilon \leq 2/3$. For $t \in 2^{I_0 \cup \cdots \cup I_{n-1}}$, we say that t is good if $t{\restriction}I_0 \notin U_0^\emptyset$, and for $j \leq n-1$, $t{\restriction}I_j \notin U_j^{t\restriction I_0 \cup \cdots \cup I_{j-1}}$. In particular, if $t \in 2^{I_0 \cup \cdots \cup I_{n-1}}$ is good, then by induction we show that

$$\mu(U_n^t) \leq \frac{2^{n+2}}{2^{n+2}-1} \cdot \frac{\mu(U \cap [t])}{\mu([t])} \leq \frac{2^{n+2}}{2^{n+2}-1} \cdot \left(1 - \frac{1}{2^{n+1}}\right) \leq \frac{2^{n+2}-2}{2^{n+2}-1}.$$

For a good sequence $t \in 2^{I_0 \cup \cdots \cup I_{n-1}}$, let

$$Z_n^t = \{v \in 2^{I_n} : \exists i \in 2 a_n^i + v \subseteq U_n^t\}.$$

By Theorem 3(1), for $i = 0, 1$ and $X \subseteq 2^{I_n}$ of size $n + 10$,

$$\frac{|X + a_n^i|}{2^{|I_n|}} = 1 - \frac{|\bigcap_{x \in X}(a_n^{1-i} + x)|}{2^{|I_n|}} \geq 1 - \frac{1}{2^{n+10}} - \frac{1}{4^{n+3}} \geq \frac{2^{n+2}-2}{2^{n+2}-1}.$$

Therefore, $|Z_n^t| \leq n + 10$. Let $Z_n = \bigcup\{Z_n^t : t \in 2^{I_0 \cup \cdots \cup I_{n-1}} \text{ is good}\}$. We get that $|Z_n| \leq (n+10) \cdot 2^{|I_0 \cup \cdots \cup I_{n-1}|}$, so in particular, $|Z_n| \cdot 2^{-|I_n|} \leq 2^{-n}$. Let

$$W_U = \{z \in 2^\omega : \exists n z{\restriction}I_n \in Z_n\}.$$

Note that if $\varepsilon < 2^{-n-|I_0 \cup \cdots \cup I_n|}$, then $Z_0 = Z_1 = \cdots = Z_n = \emptyset$. Therefore, $\mu(W_U) \longrightarrow 0$ as $\varepsilon \longrightarrow 0$. The following lemma finishes the proof.

LEMMA 10: *If $F^{-1}(y) \subseteq U + z$, then $z \in W_U$.*

Proof: Suppose not. By induction build a branch $r \in z + F^{-1}(y)$ such that $r + z{\restriction}I_0 \notin (U)_0^{\emptyset}$, $r + z{\restriction}I_1 \notin (U)_1^{r+z{\restriction}I_0}$, etc. Since U is open, it means that $r + z \notin U$. ∎

4. Main result

THEOREM 11: *Assume that $\mathsf{cov}(\mathcal{N}) = 2^{\aleph_0}$. There exists a set $X \in \mathcal{SM}$ and a continuous function $F\colon 2^\omega \longrightarrow 2^\omega$ such that $F``(X) = 2^\omega$.*

Suppose that we have sequence $\{\epsilon_n : n \in \omega\}$ and a partition $\{I_n : n \in \omega\}$ such that

(1) $2^{|I_0 \cup I_1 \cup \cdots \cup I_n|} \cdot \epsilon_{n+1} < \epsilon_n^2 < 2^{-2n}$,

(2) I_n, $a_n^0 \dot\cup a_n^1 = 2^{I_n}$ is chosen for ϵ_n as in Theorem 7.

Note that $\varepsilon_n = \sqrt{\epsilon_n}$ for $n \in \omega$ will satisfy the conclusion of Theorem 7.

Define $F\colon 2^\omega \longrightarrow 2^\omega$ as $F(x)(n) = i \iff x{\restriction}I_n \in a_n^i$ for $n \in \omega$.

The following lemma gives an abstract condition for our construction to work.

LEMMA 12: *Suppose that a σ-ideal $\mathcal{I}$ on 2^ω, and σ-ideals $\mathcal{J}_x$ of $F^{-1}(x)$ are such that the following holds:*

(1) *for every $G \in \mathcal{N}$, $\{z \in 2^\omega : \exists x(G + z) \cap F^{-1}(x) \notin \mathcal{J}_x\} \in \mathcal{I}$,*

(2) *for every $G \in \mathcal{N}$ and $t \in 2^\omega$, $\{z : t \in (G + z)\} \in \mathcal{I}$.*

(3) $\forall x\, \mathsf{cov}(\mathcal{J}_x) = \mathsf{cov}(\mathcal{I}) = 2^{\aleph_0}$.

Then there exists a set $\in \mathcal{SM}$ such that $F``(X) = 2^\omega$.

Proof: Let $\{G_\alpha : \alpha < c\}$ be enumeration of null sets, and $\{t_\alpha : \alpha < c\}$ enumeration of 2^ω. Build by induction sequences $\{x_\alpha, z_\alpha : \alpha < c\}$ such that

(1) $x_\alpha \in F^{-1}(t_\alpha)$,

(2) $\forall \beta < c\ x_\beta \notin G_\alpha + z_\alpha$.

Suppose that $\{x_\beta, z_\beta : \beta < \alpha\}$ are given.

Consider sets $Z = \{z \in 2^\omega : \exists x(G_\alpha + z) \cap F^{-1}(x) \notin \mathcal{J}_x\}$, and for $\beta < \alpha$, $Z_\beta = \{z : x_\beta \in (G_\alpha + z)\}$. Let $z_\alpha \notin Z \cup \bigcup_{\beta<\alpha} Z_\beta$. Next consider $F^{-1}(t_\alpha)$ and choose $x_\alpha \in F^{-1}(t_\alpha) \setminus \bigcup_{\beta\leq\alpha}(G_\beta + z_\beta)$. Notice that $\forall_{\beta<\alpha} F^{-1}(t_\alpha) \cap (G_\beta + z_\beta) \in \mathcal{J}_x$. ∎

In our case $\mathcal{I} = \mathcal{N}$ and $\mathcal{J}_x$ is the measure ideal on $F^{-1}(x)$. In other words, let $c_n = 2^{|I_n|-1}$, and let $\mathcal{J}$ be a σ-ideal of null sets (with respect to the standard product measure) on $\mathcal{X} = \prod_{n=0}^\infty c_n$. Note that $\mathcal{X}$ is chosen to be isomorphic (level by level) with $F^{-1}(x)$, for any x. Let $\mathcal{J}_x$ be the copy of $\mathcal{J}$ on $F^{-1}(x)$.

Specifically, define measure μ_x on $F^{-1}(x)$ as $\mu_x = \prod_{n\in\omega} \mu_n^{x(n)}$, where μ_n^i is a normalized counting measure on a_n^i $(i = 0, 1)$. Clearly, μ_x is essentially the Lebesgue measure on $F^{-1}(x)$.

Now what we want to show is that

LEMMA 13: *For every $G \in \mathcal{N}$,*

$$\{z : \exists x \mu_x(F^{-1}(x) \cap (G + z)) > 0\} \in \mathcal{N}.$$

Before we go further let us briefly look at the nature of the difficulties in proving the result using Theorem 9. The problem is that the relation $F^{-1}(y) \not\subseteq G + z$ is not additive. Quick analysis shows that every choice of a point z that will shift G away from the set we are constructing has to fulfill continuum many requirements. This is why we change from $F^{-1}(y) \not\subseteq G+z$ to $F^{-1}(y)\cap(G+z) \in \mathcal{J}_y$, an additive requirement. We still have continuum many constraints; this time we need to find z such that $F^{-1}(y) \cap (G + z) \in \mathcal{J}_y$, for *every* y.

5. Proof of Lemma 13

Suppose that $G \subseteq 2^\omega$ is a null set.

LEMMA 14: *There are sequences $\{K_n, K'_n : n \in \omega\}, \{J_n, J'_n : n \in \omega\}$ such that*

(1) *K_n's and K'_n's are consecutive intervals that are unions of I_m's,*
(2) $J_n \subseteq 2^{K_n}$, $J'_n \subseteq 2^{K'_n}$,
(3) $|J_n|/2^{|K_n|}, |J'_n|/2^{|K'_n|} < 1/2^n$,
(4) *$G \subseteq H_1 \cup H_2$, where*

$$H_1 = \{x : \exists^\infty n\ x{\restriction}K_n \in J_n\} \quad \text{and} \quad H_2 = \{x : \exists^\infty n\ x{\restriction}K'_n \in J'_n\}.$$

Proof: Use the Theorem (and its proof) 2.5.7 of [1]. ∎

Clearly, if we show Lemma 13 for H_1 and for H_2, then we show it for G. Therefore, without loss of generality we can assume that

$$G = \{x : \exists^\infty n\ x{\restriction}K_n \in J_n\},$$

where K_n, J_n are as above. Moreover, we can assume that $|J_n|/2^{|K_n|} = 1/2^n$, since the property we are interested in reflects downwards.

Now suppose that for $n \in \omega$, $K_n = I_{k_n} \cup \cdots \cup I_{k_{n+1}-1}$. Fix $z, x \in 2^\omega$ and let for $n \in \omega$,

$$J_n^{x,z} = (J_n + z{\restriction}K_n) \cap \prod_{j=k_n}^{k_{n+1}-1} a_j^{x(j)}.$$

Of course $J_n^{x,z}$ depends only on $z\restriction K_n$ and $x\restriction[k_n, k_{n+1})$.

It is easy to see that $F^{-1}(x)\cap(G+z)=\{v\in F^{-1}(x):\exists^\infty n v\restriction K_n\in J_n^{x,z}\}$. Since $F^{-1}(x)=\prod_n\prod_{j=k_n}^{k_{n+1}-1}a_j^{x(j)}$, it follows that

$$\mu_x(F^{-1}(x)\cap(G+z))=0 \iff \sum_n \frac{|J^{x,z}|}{|\prod_{j=k_n}^{k_{n+1}-1}a_j^{x(j)}|}<\infty.$$

Thus we need to find sets $T_n\subseteq 2^{K_n}$ such that $\mu(\prod_n T_n)>0$ and such that if $z\in\mathbb{Q}+\prod_n T_n$, then

$$\forall x\in 2^\omega \sum_n \frac{|J^{x,z}|}{|\prod_{j=k_n}^{k_{n+1}-1}a_j^{x(j)}|}<\infty.$$

Fix $n\in\omega$ and let $K_n=K=I_k\cup I_{k+1}\cup\cdots\cup I_{k+n}$, $J\subseteq 2^K$, $|J|/2^{|K|}=2^{-k}\geq\varepsilon_k$. It suffices to show that there exists a set $T_J\subseteq 2^K$ such that $|T_J|/2^{|K|}>1-\epsilon_{k-1}$ and for every $s\in T_J$, for every $t\in 2^{[k,k+n]}$,

$$\left|\frac{|(J+s)\cap\prod_{j=0}^n a_{k+j}^{t(j)}|}{|\prod_{j=0}^n a_{k+j}^{t(j)}|}-\frac{|J|}{2^{|K|}}\right|<\epsilon_{k-1}.$$

In this way for $z\in\mathbb{Q}+\prod_n T_n$, and $x\in 2^\omega$, and sufficiently large n,

$$\frac{|J_n^{x,z}|}{|\prod_{j=k_n}^{k_{n+1}-1}a_j^{x(j)}|}=\frac{|(J+z)\cap\prod_{j=0}^n a_{k+j}^{x(j)}|}{|\prod_{j=0}^n a_{k+j}^{t(j)}|}\leq\frac{|J|}{2^{|K|}}+\epsilon_{k-1}.$$

It follows that to finish the construction it suffices to prove the lemma below.

Lemma 15: *Suppose that $K=I_k\cup I_{k+1}\cup\cdots\cup I_{k+n}$, $J\subseteq 2^K$, $|J|/2^{|K|}\geq\varepsilon_k$. Then there exists a set $T_J\subseteq 2^K$ such that $|T_J|/2^{|K|}>1-\epsilon_{k-1}$ and for every $s\in T_J$, and every $t\in 2^{[k,k+n]}$,*

$$\left|\frac{|(J+s)\cap\prod_{j=0}^n a_{k+j}^{t(j)}|}{|\prod_{j=0}^n a_{k+j}^{t(j)}|}-\frac{|J|}{2^{|K|}}\right|<\epsilon_{k-1}.$$

Proof: For $0\leq i\leq n$, define distribution m_i on $2^{I_k\cup I_{k+1}\cup\cdots\cup I_{k+i}}$ as

$$m_i(s)=\frac{|\{t\in J:s\subseteq t\}|}{2^{|K|}}.$$

Note that $\overline{m_i}=|J|/2^{|K|}$. Observe that the distribution m_n coincides with J, that is,

$$m_n(s)=\begin{cases}1/2^{|K|} & \text{if } s\in J,\\ 0 & \text{otherwise.}\end{cases}$$

We will show by induction that for $i \leq n$, there exists a set $T_{m_i} \subseteq 2^{I_k \cup I_{k+1} \cup \cdots \cup I_{k+i}}$,

$$\frac{|T_{m_i}|}{2^{|I_k \cup I_{k+1} \cup \cdots I_{k+i}|}} > (1-\epsilon_k) \cdot \prod_{j<i}(1-\epsilon_{k+j}) > 1-\epsilon_{k-1}$$

such that for every $s \in T_{m_i}$,

$$\left| \sum \left\{ m_i(t) : t \in \prod_{j=0}^{i} (a^0_{k+j} + s{\restriction}I_{k+j}) \right\} - \frac{1}{2^{i+1}} \cdot \frac{|J|}{2^{|K|}} \right| < \frac{\epsilon_{k-1}}{2^{i+1}}.$$

In particular, for $i=n$, and $s \in T_{m_n} = T_J$,

$$\left| \sum \left\{ m_n(t) : t \in \prod_{j=0}^{n} (a^0_{k+j} + s{\restriction}I_j) \right\} - \frac{1}{2^{n+1}} \cdot \frac{|J|}{2^{|K|}} \right| < \frac{\epsilon_{k-1}}{2^{n+1}}.$$

The last equation means that for $s \in T_J$,

$$\left| \frac{|J \cap (\prod_{j=0}^{n} a^0_{k+j} + s{\restriction}I_j)|}{2^{|K|}} - \frac{1}{2^{n+1}} \cdot \frac{|J|}{2^{|K|}} \right| < \frac{\epsilon_{k-1}}{2^{n+1}}.$$

By moving s, and multiplying by 2^{n+1}, we finally get

$$\left| \frac{|(J+s) \cap \prod_{j=0}^{n} a^0_{k+j}|}{|\prod_{j=0}^{n} a^0_{k+j}|} - \frac{|J|}{2^{|K|}} \right| < \epsilon_{k-1}.$$

Let

$$\left| \frac{|(J+s) \cap \prod_{j=0}^{n} a^0_{k+j}|}{|\prod_{j=0}^{n} a^0_{k+j}|} - \frac{|J|}{2^{|K|}} \right| = \text{error}_n .$$

We want to show that $\text{error}_n < \epsilon_{k-1}$.

Before we start induction note that we can shrink J slightly, so that the resulting set has the following property,

$$\forall i \leq n \forall s \in 2^{I_k \cup \cdots \cup I_{k+i}} \left(m_i(s) \neq 0 \to m_i(s) \geq \frac{\varepsilon_{k+i}}{2^{|I_k \cup \cdots \cup I_{k+i}|}} \right).$$

By removing from J all nodes (and their descendants) that do not have this property we drop the "measure" by $\varepsilon_{k+n} + \varepsilon_{k+n-1} + \cdots + \varepsilon_k < 2\varepsilon_k < \epsilon_{k-1}/3$. So let assume that J has the above property and later add $\epsilon_{k-1}/3$ to the error term.

The inductive proof is straightforward — for m_0 we get T_{m_0} immediately from Theorem 7.

Now consider m_{i+1}. For each $t \in 2^{I_k \cup \cdots \cup I_{k+i}}$, let m^t_{i+1} be the distribution on $2^{I_{k+i+1}}$ defined as $m^t_{i+1}(s) = 2^{|I_k \cup \cdots \cup I_{k+i}|} m_{i+1}(t^\frown s)$.

Clearly,

$$m^t_{i+1}(s) \leq 2^{|I_k \cup \cdots \cup I_{k+i}|} \frac{1}{2^{|I_k \cup \cdots \cup I_{k+i+1}|}} = \frac{1}{2^{|I_{k+i+1}|}}, \quad \text{for every } s.$$

Moreover, $\overline{m^t_{i+1}} = 2^{|I_k \cup \cdots \cup I_{k+i}|} m_i(t)$. In particular, shrinking J as above yields, if $\overline{m^t_{i+1}} > 0$, then $\overline{m^t_{i+1}} \geq \varepsilon_{k+i}$. For every $t \in 2^{I_k \cup \cdots \cup I_{k+i}}$, $\overline{m^t_{i+1}} > 0$ apply Theorem 7 to get a set $T_t \subseteq 2^{I_{k+i+1}}$ such that $|T_t|/2^{|I_{k+i+1}|} \geq 1 - \epsilon_{k+i+1}$, and for every $s \in T_t$,

$$\left| \sum \{ m^t_{i+1}(v) : v \in a^0_{k+i+1} + s \} - \overline{m^t_{i+1}}/2 \right| < \epsilon_{k+i+1}.$$

Let $T_{m_{i+1}} = T_{m_i} \times \bigcap \{T_t : \overline{m^t_{i+1}} > 0\}$. Clearly,

$$\begin{aligned}
\frac{|T_{m_{i+1}}|}{2^{|I_k \cup \cdots \cup I_{k+i+1}|}} =& \frac{|T_{m_i}|}{2^{|I_k \cup \cdots \cup I_{k+i}|}} \cdot \frac{|\bigcap_t T_t|}{2^{|I_{k+i+1}|}} \\
\geq & ((1-\epsilon_k) \cdot \prod_{j<i} (1 - \epsilon_{k+j})) \cdot (1 - 2^{|I_k \cup \cdots \cup I_{k+i+1}|} \epsilon_{k+i+1}) \\
\geq & (1-\epsilon_k) \cdot \prod_{j \leq i} (1 - \epsilon_{k+j}) > 1 - \epsilon_{k-1}.
\end{aligned}$$

Suppose that $s \in T_{m_{i+1}}$.

$$\begin{aligned}
& \sum \left\{ m_{i+1}(v) : v \in \prod_{j=0}^{i+1} (a^0_{k+j} + s \restriction I_{k+j}) \right\} \\
=& \sum \left\{ \sum \{ m_{i+1}(t^\frown v) : v \in a^0_{k+i+1} + s \restriction I_{k+i+1} \} : t \in \prod_{j=0}^{i} (a^0_{k+j} + s \restriction I_{k+j}) \right\} \\
=& \sum \left\{ \sum \left\{ \frac{1}{2^{|I_k \cup \cdots \cup I_{k+i}|}} m^t_{i+1}(v) : v \in a^0_{k+i+1} + s \restriction I_{k+i+1} \right\} : \right. \\
& \left. t \in \prod_{j=0}^{i} (a^0_{k+j} + s \restriction I_{k+j}) \right\} \\
\leq & \sum \left\{ \frac{1}{2^{|I_k \cup \cdots \cup I_{k+i}|}} \left(\frac{\overline{m^t_{i+1}}}{2} + \epsilon_{k+i+1} \right) : t \in \prod_{j=0}^{i} (a^0_{k+j} + s \restriction I_{k+j}) \right\} \\
\leq & \frac{1}{2} \sum \left\{ \frac{1}{2^{|I_k \cup \cdots \cup I_{k+i}|}} (\overline{m^t_{i+1}} + 2\epsilon_{k+i+1}) : t \in \prod_{j=0}^{i} (a^0_{k+j} + s \restriction I_{k+j}) \right\} \\
=& \frac{1}{2} \sum \left\{ m_i(t) + \frac{2\epsilon_{k+i+1}}{2^{|I_k \cup \cdots I_{k+i}|}} : t \in \prod_{j=0}^{i} (a^0_{k+j} + s \restriction I_{k+j}) \right\} \\
\leq & \epsilon_{k+i+1} + \frac{1}{2} \left(\frac{1}{2^{i+1}} \frac{|J|}{2^{|K|}} + \text{error}_i \right) \leq \frac{1}{2^{i+2}} \frac{|J|}{2^{|K|}} + \frac{\text{error}_i}{2} + \epsilon_{k+i+1},
\end{aligned}$$

where error_i is the error term given by the inductive hypothesis. That gives us

$$\text{error}_i \leq \frac{\epsilon_k}{2^i} + \frac{\epsilon_{k+1}}{2^{i-1}} + \cdots + \epsilon_{k+i} \leq \frac{\epsilon_{k-1}}{2^{i+2}},$$

so

$$\sum\left\{m_{i+1}(v) : v \in \prod_{j=0}^{i+1}(a^0_{k+j} + s\restriction I_{k+j})\right\} \leq \frac{1}{2^{i+2}}\frac{|J|}{2^{|K|}} + \frac{\epsilon_{k-1}}{2^{i+3}} + \epsilon_{k+i+1}.$$

The lower bound is similar, and we get for $s \in T_{m_{i+1}}$,

$$\left|\sum\left\{m_{i+1}(t) : t \in \prod_{j=0}^{i+1}(a^0_{k+j} + s\restriction I_{k+j})\right\} - \frac{1}{2^{i+2}} \cdot \frac{|J|}{2^{|K|}}\right| < \frac{\epsilon_{k-1}}{2^{i+3}} + \epsilon_{k+i+1}.$$

As before that yields the estimate

$$\left|\frac{|(J+s) \cap \prod_{j=0}^n a^0_{k+j}|}{|\prod_{j=0}^n a^0_{k+j}|} - \frac{|J|}{2^{|K|}}\right| < \frac{\epsilon_{k-1}}{2} + 2^{i+2} \cdot \epsilon_{k+i+1}.$$

Since we started by reducing the "measure" of J by $\epsilon_{k-1}/3$ we get the required estimate.

Finally we will show the second part of the lemma. We will proceed by induction on n. If $n = 0$, then $K = I_k$ and by the part already proved

$$\left|\frac{|(J+s) \cap a^0_k|}{|a^0_k|} - \frac{|J|}{2^{|K|}}\right| < \epsilon_{k-1}.$$

Now

$$\begin{aligned}|(J+s) \cap a^1_k| = |(J+s) \setminus ((J+s) \cap a^0_k)| &\leq |J| - \left(\frac{|J|}{2^{|K|}} - \epsilon_{k-1}\right) \cdot |a^0_k| \\ &\leq |J+s| - \frac{1}{2}|J| + \epsilon_{k-1}|a^0_k| = \frac{1}{2}|J| + \epsilon_{k-1}|a^0_k|.\end{aligned}$$

Thus

$$\frac{|(J+s) \cap a^1_k|}{2^{|K|}} \leq \frac{\frac{1}{2}|J| + \epsilon_{k-1}|a^0_k|}{2^{|K|}} = \frac{1}{2}\frac{|J|}{2^{|K|}} + \frac{\epsilon_{k-1}}{2}.$$

The lower estimate is similar, so we have

$$\left|\frac{|(J+s) \cap a^1_k|}{2^{|K|}} - \frac{1}{2}\frac{|J|}{2^{|K|}}\right| < \frac{\epsilon_{k-1}}{2},$$

and

$$\left|\frac{|(J+s) \cap a^1_k|}{|a^1_k|} - \frac{|J|}{2^{|K|}}\right| < \epsilon_{k-1}.$$

The rest of the proof is a repetition of the above argument; the single step computed here shows that there is no difference whether we use a^0_j or a^1_j, the estimates do not change. ∎

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